Approximate analytical solutions of the Dirac equation for Yukawa potential plus Tensor Interaction with any κ-value

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Abstract

Approximate analytical solutions of the Dirac equation are obtained for the Yukawa potential plus a tensor interaction with any κ-value for the cases having the Dirac equation pseudospin and spin symmetry. The potential describing tensor interaction has a Yukawa-like form. Closed forms of the energy eigenvalue equations and the spinor wave functions are computed by using the Nikiforov-Uvarov method. It is observed that the energy eigenvalue equations are consistent with the ones obtained before. Our numerical results are also listed to see the effect of the tensor interaction on the bound states.

Keywords: Yukawa potential, Dirac equation, Nikiforov-Uvarov method, spin symmetry, pseudospin symmetry

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I. INTRODUCTION

The Yukawa potential has the form [1]

$$V(r) = -\frac{\eta}{r}\, e^{-\beta r}\,. \tag{1}$$

Here β is the screening parameter and η is the potential strength. This potential is one of the different types of the screened Coulomb potentials which have been studied in various areas of physics such as atomic physics, plasma physics, solid-state and astrophysics [2]. The readers can find an outline about the approaches used to study this potential in Ref. [3]. The Yukawa potential is a basic ground to describe interaction between charged particles in colloidal suspensions [4] and an important potential model in Thomas-Fermi approximation to the electron gas [5]. In the view of the present work, it should be noted that, with its other properties, the Yukawa potential has bound states only for the values of the parameter β below a value $\beta_c < 1.19$ (in a.u.) [6]. The Yukawa potential [7-15] has also received a great deal of attention in view of the methods by which the potential has been studied. Some of them used to solving the wave equations are the $1/N$-expansion [16] and shifted $1/N$-expansion methods [17], studying the potential by using the Raygleih-Schrödinger perturbation expansion [18], a group-theoretical approach by using the Fock transformation [3], the variational self-consistent field molecular-orbital method [18], the J-matrix method [19], a new approximation scheme proposed to study the bound states of potential [20], studying in terms of the hypervirial theorems [21] and a numerical solution of the Schrödinger equation for the present potential [22].

In the present work, we study the problem including also the two-component spinor wave functions in terms of the hypergeometric functions within the context of pseudospin and spin symmetries [23-25]. The Dirac equation with vector, $V(r)$, and scalar, $S(r)$, potentials has pseudospin (spin) symmetry when the difference (the sum) of the potentials $V(r) - S(r)$ $[V(r) + S(r)]$ is constant, which means $\frac{d}{dr}[V(r) - S(r)] = 0$ (or $\frac{d}{dr}[V(r) + S(r)] = 0$). It is pointed out that these symmetries can explain degeneracies in single-particle energy levels in nuclei or in some heavy meson-spectra within the context of relativistic mean-field theories [23-35]. In the relativistic domain, these symmetries were used in the context of deformation and superdeformation in nuclei, magnetic moment interpretation and identical bands [26]. In the non-relativistic domain, performing a helicity unitary transformation to a single-particle Hamiltonian maps the normal state onto the pseudo-state [27]. Because of these investigations, the solutions of the Dirac equation having spin and pseudospin symmetry have received great attention for different type of potentials such as Morse potential,

Eckart potential, etc. [28-32].

In Ref. [33] the pseudospin symmetric solutions of the Dirac equation are obtained for the harmonic oscillator while used the tensor interaction as a potential linear in r. In Ref. [34] spin and pseudospin symmetric solutions are studied for the Woods-Saxon potential by taking the tensor interaction as a Coulomb-like potential. In this point of view, we propose to use a Yukawa-like potential as the tensor interaction

$$U(r) = \nu \frac{e^{-\beta r}}{r}\,, \tag{2}$$

which has an attractive form. Choosing this form makes it possible to find analytical solutions of the present problem.

The organization of this work is as follows. In Section 2, we briefly give the Dirac equation with attractive scalar and repulsive vector potentials for the cases where the Dirac equation has pseudospin and spin symmetry, respectively. In Section 3, we present the Nikiforov-Uvarov (NU) method and the parameters required within the method. In Section 4, we find an analytical expression for the bound states and the two-component spinor wave functions of the Yukawa potential by using an approximation instead of the spin-orbit coupling term. We analyze the problem for the cases having the Dirac equation pseudospin and also spin symmetry. We give the numerical energy eigenvalues for the different quantum number pairs (n, κ) where we choose the parameter and the mass values in $a.u.$. The last section includes our conclusions.

II. DIRAC EQUATION AND SPIN AND PSEUDOSPIN SYMMETRY

The Dirac equation is basically written by using linear momentum operator, $P_\mu = i\hbar\partial_\mu$ (four-vector) and the scalar rest mass M. As a result, two potential couplings are used in equation. One coupling is a gauge invariant one to the four-vector potential $A_\mu(\vec{r}, t)$ by using $P_\mu \to P_\mu - gA_\mu$ (g is a real coupling parameter) and the other one is to the space-time scalar potential $S(\vec{r}, t)$ by substitution $M \to M + S$. The "four-vector" and "scalar" terms mean that classifying the observable according to the unitary irreducible representation of the rotation and translation groups in Minkowski space-time. By taking the space component of the vector potential to vanish ($\vec{A} = 0$) and writing the time component of the four-vector potential as $gA_0 = V(\vec{r}, t)$, then we obtain the so-called vector and scalar potentials, $V(r)$ and $S(r)$, respectively.

The free particle Dirac equation is given ($\hbar = c = 1$)

$$\big(i\gamma^\mu \partial_\mu - M\big)\Psi(\vec{r}, t) = 0\,, \tag{3}$$

and taking the total wave function as $\Psi(\vec{r},t) = e^{-iEt}\psi(\vec{r})$ for time-independent potentials, where E is the relativistic energy, the above equation including also a tensor interaction, $U(r)$, with spherical symmetric vector and scalar potentials is written as

$$\left[\vec{\alpha}.\vec{P} + \beta M - i\beta\vec{\alpha}.\hat{r}U(r) + \beta S(r)\right]\psi(\vec{r}) = \left[E - V(r)\right]\psi(\vec{r})\,, \tag{4}$$

Here α and β are usual 4×4 matrices. For spherical nuclei, the angular momentum $\vec{J}$ and the operator $\hat{K} = -\beta\left(\hat{\sigma}.\hat{L}+1\right)$ with eigenvalues $\kappa = \pm(j+1/2)$ commute with the Dirac Hamiltonian, where $\hat{L}$ is the orbital angular momentum. By using the radial eigenfunctions for upper and lower components of the Dirac eigenfunction $F(r)$ and $G(r)$, respectively, the wave function is written as [35]

$$\psi(\vec{r}) = \frac{1}{r}\begin{bmatrix} F\,(r)Y^{(1)}(\theta,\phi) \\ iG\,(r)Y^{(2)}(\theta,\phi) \end{bmatrix}, \tag{5}$$

where $Y^{(1)}(\theta,\phi)$ and $Y^{(2)}(\theta,\phi)$ are the pseudospin and spin spherical harmonics, respectively. They correspond to angular and spin parts of the wave function given by

$$Y^{(1),(2)}(\theta,\phi) = \sum_{m_\ell m_s} < \ell m_\ell \frac{1}{2} m_s | \ell \frac{1}{2} j m > Y_{\ell m_\ell}(\theta,\phi)\chi_{\frac{1}{2}m_s}\,,$$
$$j = |\kappa| - \frac{1}{2}\,,\;\; \ell = \kappa\ (\kappa > 0)\,;\, \ell = -(\kappa+1)\ (\kappa < 0)\,, \tag{6}$$

Here, $Y_{\ell m_\ell}(\theta,\phi)$ denotes the spherical harmonics and m_ℓ and m_s are related magnetic quantum numbers.

Substituting Eq. (5) into Eq. (4) and using the followings

$$\left(\vec{\sigma}.\vec{L}\right)Y^{(2)}(\theta,\phi) = (\kappa-1)Y^{(2)}(\theta,\phi)\,, \tag{7a}$$

$$\left(\vec{\sigma}.\vec{L}\right)Y^{(1)}(\theta,\phi) = -(\kappa-1)Y^{(1)}(\theta,\phi)\,, \tag{7b}$$

$$\left(\vec{\sigma}.\hat{r}\right)Y^{(2)}(\theta,\phi) = -Y^{(1)}(\theta,\phi)\,, \tag{7c}$$

$$\left(\vec{\sigma}.\hat{r}\right)Y^{(1)}(\theta,\phi) = -Y^{(2)}(\theta,\phi)\,, \tag{7d}$$

give us the following coupled differential equations

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F(r) = [E + M - \Gamma(r)]G(r)\,, \tag{8a}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G(r) = [M - E + \Lambda(r)]F(r)\,. \tag{8b}$$

where $\Gamma(r) = V(r) - S(r)$ and $\Lambda(r) = V(r) + S(r)$. Using the expression $G(r)$ in Eq. (8a) and inserting it into Eq. (8b), we get a second order differential equation

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \varepsilon^{(1)}(r) + \left(\frac{2\kappa}{r} - U(r) - \frac{d}{dr}\right)U(r)\right]F(r) = -\left[\frac{d\Gamma(r)/dr}{[E + M - \Gamma(r)]}\right]F(r)\,, \tag{9}$$

where $\varepsilon^{(1)}(r) = [E+M-\Gamma(r)]\,[E-M-\Lambda(r)]$. By similar steps, we write the following second order differential equation for $G(r)$ as

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \varepsilon^{(2)}(r) + (\frac{2\kappa}{r} - U(r) + \frac{d}{dr})\right] G(r) = \left[\frac{d\Lambda(r)/dr}{[M-E+\Lambda(r)]}\right] G(r)\,, \tag{10}$$

where $\varepsilon^{(2)}(r) = [E-M-\Lambda(r)]\,[E+M-\Gamma(r)]$. The last two equations have the following forms

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + (\frac{2\kappa}{r} - U(r) - \frac{d}{dr})U(r) + [E+M-A]\,[E-M-\Lambda(r)]\right\} F(r) = 0\,, \tag{11a}$$

$$\left\{\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + (\frac{2\kappa}{r} - U(r) + \frac{d}{dr})U(r) + [E-M-A]\,[E+M-\Gamma(r)]\right\} G(r) = 0\,. \tag{11b}$$

if the Dirac equation has spin symmetry which means that $\Gamma(r) = A$ $(d\Gamma(r)/dr = 0)$ is a constant and pseudospin symmetry which means $\Lambda(r) = A$ $(d\Lambda(r)/dr = 0)$ is a constant [25-27].

III. NIKIFOROV UVAROV METHOD

The Nikiforov-Uvarov method could be used to solve a second-order differential equation of the hypergeometric-type which can be transformed by using appropriate coordinate transformation into the following form

$$\sigma^2(t)\frac{d^2\Psi(t)}{dt^2} + \sigma(t)\tilde{\tau}(t)\frac{d\Psi(t)}{dt} + \tilde{\sigma}(t)\Psi(t) = 0\,, \tag{12}$$

where $\sigma(t)$, and $\tilde{\sigma}(t)$ are polynomials, at most, second degree, and $\tilde{\tau}(t)$ is a first-degree polynomial. By taking the solution as

$$\Psi(t) = \psi(t)\varphi(t)\,, \tag{13}$$

gives Eq. (12) as a hypergeometric type equation [36]

$$\frac{d^2\varphi(t)}{dt^2} + \frac{\tau(t)}{\sigma(t)}\frac{d\varphi(t)}{dt} + \frac{\lambda}{\sigma(t)}\varphi(t) = 0\,, \tag{14}$$

where $\psi(t)$ is defined by using the equation [36]

$$\frac{1}{\psi(t)}\frac{d\psi(t)}{dt} = \frac{\pi(t)}{\sigma(t)}\,, \tag{15}$$

and the other part of the solution in Eq. (13) is given by

$$\varphi_n(t) = \frac{a_n}{\rho(t)}\frac{d^n}{dz^n}[\sigma^n(z)\rho(t)]\,, \tag{16}$$

where a_n is a normalization constant, and $\rho(t)$ is the weight function, and satisfies the following equation [36]

$$\frac{d\sigma(t)}{dt} + \frac{\sigma(t)}{\rho(t)}\frac{d\rho(t)}{dt} = \tau(t)\,. \tag{17}$$

The function $\pi(t)$ and the parameter λ in the above equation are defined as

$$\pi(t) = \frac{1}{2}[\frac{d}{dt}\sigma(t) - \tilde{\tau}(t)] \pm \left\{\frac{1}{4}\left[\frac{d}{dt}\sigma(t) - \tilde{\tau}(t)\right]^2 - \tilde{\sigma}(t) + k\sigma(t)\right\}^{1/2}, \tag{18}$$

$$\lambda = k + \frac{d}{dt}\pi(t). \tag{19}$$

In the NU method, the square root in Eq. (18) must be the square of a polynomial, so the parameter k can be determined. Thus, a new eigenvalue equation becomes

$$\lambda = \lambda_n = -n\frac{d}{dt}\tau(t) - \frac{1}{2}(n^2 - n)\frac{d^2}{dt^2}\sigma(t). \tag{20}$$

where prime denotes the derivative, and the derivative of the function $\tau(t) = \tilde{\tau}(t) + 2\pi(t)$ should be negative.

IV. BOUND STATE SOLUTIONS

1. Pseudospin Symmetric Case

Taking the scalar and vector potentials in Eq. (4) as

$$S(r) = -\frac{\eta_s}{r}e^{-\beta r}\ ;\ \ V(r) = +\frac{\eta_v}{r}e^{-\beta r}, \tag{21}$$

inserting them into Eq. (11b), using Eq. (2) and taking the following expression instead of the spin-orbit coupling term [37, 38, 39]

$$\frac{1}{r^2} \simeq \beta^2\frac{1}{(1 - e^{-\beta r})^2} \tag{22}$$

we obtain from Eq. (11b)

$$\begin{aligned} &\left\{\frac{d^2}{dr^2} - \frac{\beta^2\kappa(\kappa-1)}{(1-e^{-\beta r})^2} + \left[(2\kappa\nu\beta^2 - \nu\beta^2)e^{-\beta r} - \nu^2\beta^2 e^{-2\beta r}\right]\frac{1}{(1-e^{-\beta r})^2}\right. \\ &\left. -\nu\beta^2\frac{e^{-\beta r}}{1-e^{-\beta r}} - \beta\eta_1\left[E - M - A\right]\frac{e^{-\beta r}}{1-e^{-\beta r}} + \left[E - M - A\right]\left[E + M\right]\right\}G(r) = 0, \end{aligned} \tag{23}$$

where $\eta_1 = \eta_s + \eta_v$. Using a new variable $t = 1 - e^{-2\beta r}$ $(0 \leq t \leq 1)$ and the following abbreviations

$$-a_1^2 = -\kappa(\kappa-1) + \nu(2\kappa - \nu - 1), \tag{24a}$$

$$-a_2^2 = \nu(-2\kappa + 2\nu) - \frac{\eta_1}{\beta}(E - M - A), \tag{24b}$$

$$-a_3^2 = \nu(-\nu + 1) + \frac{\eta_1}{\beta}(E - M - A) + \frac{1}{\beta^2}(E - M - A)(E + M). \tag{24c}$$

gives us

$$\left\{\frac{d^2}{dt^2}-\frac{t}{t(1-t)}\frac{d}{dt}+\frac{1}{t^2(1-t)^2}[-a_1^2-a_2^2t-a_3^2t^2]\right\}G(t)=0\,, \tag{25}$$

Comparing Eq. (25) with Eq. (12)

$$\tilde{\tau}(t)=-t\,,\quad \sigma(t)=t(1-t)\,,\quad \tilde{\sigma}(z)=-a_3^2t^2-a_2^2t-a_1^2\,, \tag{26}$$

Substituting this into Eq. (18), we get

$$\pi(t)=\frac{1-t}{2}\pm\sqrt{(\frac{1}{4}+a_3^2-k)t^2+(-\frac{1}{2}+a_2^2+k)t+\frac{1}{4}+a_1^2}. \tag{27}$$

The constant k can be determined by the condition that the discriminant of the expression under the square root has to be zero

$$\Delta=(-\frac{1}{2}+a_2^2+k)^2-4(\frac{1}{4}+a_1^2)(\frac{1}{4}+a_3^2-k)=0\,. \tag{28}$$

The roots of k are $k_{1,2}=-a_2^2-2a_1^2\mp A$, where $A=\sqrt{(a_3^2+a_2^2+a_1^2)(1+4a_1^2)}$. Substituting these values into Eq. (18), we obtain $\pi(t)$ for k_1 as

$$\pi(t)=\frac{1-t}{2}+\left[\left(\sqrt{\frac{1}{4}+a_1^2}+\frac{A}{2\sqrt{\frac{1}{4}+a_1^2}}\right)t-\sqrt{\frac{1}{4}+a_1^2}\right]\,, \tag{29}$$

and for k_2 as

$$\pi(t)=\frac{1-t}{2}-\left[\left(\sqrt{\frac{1}{4}+a_1^2}+\frac{A}{2\sqrt{\frac{1}{4}+a_1^2}}\right)t-\sqrt{\frac{1}{4}+a_1^2}\right]\,. \tag{30}$$

Now we find the polynomial $\tau(t)$ from $\pi(t)$ as

$$\tau(t)=1+2\sqrt{\frac{1}{4}+a_1^2}-\left(2+2\sqrt{\frac{1}{4}+a_1^2}+\frac{A}{\sqrt{\frac{1}{4}+a_1^2}}\right)\,. \tag{31}$$

so its derivative is negative. We have from Eq. (19)

$$\lambda=-a_2^2-2a_1^2-A-\frac{1}{2}-\frac{1}{2}\sqrt{1+4a_1^2}-\frac{A}{\sqrt{1+4a_1^2}}\,, \tag{32}$$

and Eq. (20) gives us

$$\lambda_n=n\left(2+\sqrt{1+4a_1^2}+\frac{2A}{\sqrt{1+4a_1^2}}\right)+n(n-1)\,. \tag{33}$$

Substituting the values of the parameters given by Eq. (24) and setting $\lambda=\lambda_n$, one can find the following

$$2\sqrt{\kappa(\kappa-1)-\frac{1}{\beta^2}\,(E-M-A)(E+M)}\;+2\sqrt{\nu(\nu-1)-\frac{1}{\beta}(E-M-A)\big(\eta_1+\frac{E+M}{\beta}\big)}$$
$$+\sqrt{(1-2\kappa)^2+4\nu(1+\nu-2\kappa)}\;+2n+1=0\,. \quad (34)$$

Thus one can find energy eigenvalues for the case of pseudospin symmetry for any κ-value in the existence of a tensor interaction. This analytical result is consistent with the ones obtained in Ref. [8] for the absence of the tensor interaction. Table I presents the numerical energy eigenvalues for different quantum number pairs (n,κ) and different values for ν which makes it possible to see the effect of the tensor interaction on the energy eigenstates. It is seen that energy values are negative as stated in the theory [25-27]. Our parameter values are as follow: $\eta_1=2.5$ a.u., $\beta=0.5$ a.u., $A=0$. We observe that the energy eigenvalues decrease while the principal quantum number increases or spin-orbit quantum number κ decreases.

In order to find the eigenfunctions, we first compute the weight function from Eq. (17)

$$\rho(t)=t^{2a_1}(1-t)^{A}\,, \quad (35)$$

and the wave function becomes

$$\varphi_{n\ell}\,(z)\sim\frac{1}{t^{2a_1}(1-t)^{A}}\,\frac{d^n}{dt^n}\,\left[t^{n+2a_1}\,(1-t)^{n+A}\right]\,. \quad (36)$$

The polynomial solutions can be written in terms of the Jacobi polynomials [40]

$$\varphi_{n\ell}\,(t)\sim P_n^{(A,\,2a_1)}\,(2t-1)\,,\quad A>-1\,,\quad a_1>-1\,. \quad (37)$$

On the other hand, the other part of the wave function is obtained from Eq. (15) as

$$\psi(t)=t^{a_1}(1-t)^{(1+A)/2}\,. \quad (38)$$

Thus, the radial eigenfunctions for the lower component of the Dirac eigenfunction take

$$G(t)\sim t^{a_1}(1-t)^{(1+A)/2}\,P_n^{(A,\,2a_1)}\,(2t-1)\,, \quad (39)$$

and the other radial component is obtained from Eq. (8a) as

$$F(t)\sim\frac{t^{a_1}(1-t)^{(1+A)/2}}{M-E+A}\Big\{\beta(1-t)\Big[\left(\frac{a_1}{t}-\frac{1+A}{2(1-t)}\right)P_n^{(A,\,2a_1)}\,(2t-1)$$
$$+\;(n+A+2a_1+1)P_{n-1}^{(A+1,\,2a_1+1)}\,(2t-1)\Big]$$
$$-\;\frac{\kappa\beta}{lnt}P_n^{(A,\,2a_1)}\,(2t-1)-\frac{\beta\nu}{lnt}(1-t)P_n^{(A,\,2a_1)}\,(2t-1)\Big\}\,. \quad (40)$$

where we have used the property of the Jacobi polynomials as $\frac{d}{dx}[P_n^{(q,\,r)}(x)]=\frac{1}{2}(n+q+r+1)P_{n-1}^{(q+1,\,r+1)}(x)$ [40].

2. Spin Symmetric Case

In this case, from Eqs. (2), (21) and (22), Eq. (11a) takes the form

$$\left\{\frac{d^2}{dr^2}-\frac{\beta^2\kappa(\kappa-1)}{(1-e^{-\beta r})^2}+\left[(2\kappa\nu\beta^2+\nu\beta^2)e^{-\beta r}-\nu^2\beta^2e^{-2\beta r}\right]\frac{1}{(1-e^{-\beta r})^2}\right.$$
$$\left.+\nu\beta^2\frac{e^{-\beta r}}{1-e^{-\beta r}}+\beta\eta_2\left[E+M-A\right]\frac{e^{-\beta r}}{1-e^{-\beta r}}+\left[E+M-A\right]\left[E-M\right]\right\}F(r)=0\,, \tag{41}$$

where $\eta_2=\eta_s-\eta_v$. Using the same variable gives

$$\left\{\frac{d^2}{dt^2}-\frac{t}{t(1-t)}\frac{d}{dt}+\frac{1}{t^2(1-t)^2}[-a_1^2-a_2^2t-a_3^2t^2]\right\}F(t)=0\,, \tag{42}$$

where

$$-a_1^2=-\kappa(\kappa-1)+\nu(2\kappa-\nu+1)\,, \tag{43a}$$
$$-a_2^2=\nu(-2\kappa+2\nu)+\frac{\eta_2}{\beta}\left(E+M-A\right), \tag{43b}$$
$$-a_3^2=-\nu(\nu+1)-\frac{\eta_2}{\beta}\left(E+M-A\right)+\frac{1}{\beta^2}\left(E+M-A\right)(E-M)\,. \tag{43c}$$

Following the same procedure we obtain the energy eigenvalues for the case of spin symmetry

$$2\sqrt{\kappa(\kappa+1)-\frac{1}{\beta^2}\left(E+M-A\right)(E-M)}+2\sqrt{\nu(\nu+1)+\frac{1}{\beta}(E+M-A)\big(\eta_2-\frac{E-M}{\beta}\big)}$$
$$+\sqrt{(1+2\kappa)^2-4\nu(1-\nu+2\kappa)}-2n-1=0\,, \tag{44}$$

and the radial eigenfunctions for the upper component of the Dirac eigenfunction as

$$F(t)\sim t^{a_1}(1-t)^{(1+A)/2}\,P_n^{(A,\,2a_1)}\left(2t-1\right), \tag{45}$$

which gives us the other radial component from Eq. (8a) as

$$G(t)\sim\frac{t^{a_1}(1-t)^{(1+A)/2}}{M-E+A}\left\{P_n^{(A,\,2a_1)}\left(2t-1\right)\left[\beta(1-t)\big(\frac{a_1}{t}-\frac{1+A}{2(1-t)}+\frac{\kappa\beta}{lnt}+\frac{\beta\nu}{lnt}(1-t)\big)\right]\right.$$
$$\left.+\beta(1-t)(n+A+2a_1+1)P_{n-1}^{(1+A,\,1+2a_1)}\left(2t-1\right)\right\}. \tag{46}$$

The numerical energy eigenvalues for different quantum number pairs (n,κ) and different parameter values which makes it possible to see the effect of the tensor interaction on the energy eigenstates are showed in Table I (for $\eta_2=2.5$ a.u., $\beta=0.5$ a.u., $A=0$). We see that energy values are positive if the Dirac equation has spin symmetry [25-27]. We observe that the energy eigenvalues increase while the principal quantum number increases or spin-orbit quantum number κ decreases. It could

be interesting to study the case if we take the tensor interaction as a Coulomb-like potential. Eq. (2) has the following form for $\beta r \to 0$

$$U(r) \sim \frac{\nu}{r}\left(1 - \beta r + \ldots\right), \tag{47}$$

which gives us an attractive Coulomb potential for $|\nu| < 0$ and a repulsive one for $|\nu| > 0$ for the first-order approximation. In this case, we obtain the energy eigenvalue equation for the case of pseudospin symmetry

$$2\sqrt{\kappa(\kappa-1) - \nu(2\kappa - 1 - \nu) - \frac{1}{\beta^2}\left[E^2 - M^2 - A(E+M)\right]} + 2\sqrt{\frac{1}{\beta}(M - E + A)\big(\eta_1 + \frac{E+M}{\beta}\big)} \\ + \sqrt{(1-2\kappa)^2 - 4\nu(2\kappa - 1 - \nu)} + 2n + 1 = 0\,. \tag{48}$$

which is valid for attractive Coulomb potential while the eigenvalue equation for the case of spin symmetry is written as

$$2\sqrt{\kappa(\kappa+1) - \nu(2\kappa + 1 - \nu) - \frac{1}{\beta^2}\left[E^2 - M^2 - A(E+M)\right]} + 2\sqrt{\frac{1}{\beta}(M + E - A)\big(\eta_2 + \frac{M-E}{\beta}\big)} \\ + \sqrt{(1+2\kappa)^2 - 4\nu(2\kappa + 1 - \nu)} - 2n - 1 = 0\,. \tag{49}$$

It should be noted that the terms including the tensor interaction in Eq. (11) behave like a centrifugal barrier if one chooses the tensor interaction as a Coulomb-like potential. So, we could except that the number of bound states increase because of the contributions coming from the tensor terms to centrifugal barrier. We summarize the numerical results in Table II for the case where if we take the tensor interaction as a Coulomb-like potential given in Eq. (47).

V. CONCLUSIONS

We have studied the approximate bound state solutions of the Dirac equation for the Yukawa potential for the cases where the Dirac equation has pseudospin and spin symmetry, respectively, in the existence of a tensor interaction having a Yukawa-like form. We have obtained the energy eigenvalue equations and the related two-component spinor wave functions with the help of Nikiforov-Uvarov method. We have presented the numerical results of the energy eigenvalues for the cases of pseudospin and spin symmetry in Table I to see the effect of tensor interaction on bound state energies and seen that this contribution is to create much strongly bound states.

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TABLE I: Energy eigenvalues in units of a.u. of the Yukawa potential for the cases of pseudospin and spin symmetry ($M = 1$).

			pseudospin symmetry		
			$\nu = 0$	$\nu = 0.1$	$\nu = 1.0$
n	κ	state	$E < 0$	$E < 0$	$E < 0$
1	-1	$1s_{1/2}$	1.22470	1.22461	1.13610
	-2	$1p_{3/2}$	1.46034	1.45258	1.23104
	-3	$1d_{5/2}$	1.60000	1.58996	1.29851
	-4	$1f_{7/2}$	1.69025	1.67938	1.34844
2	-1	$2s_{1/2}$	1.13610	1.12562	0.84186
	-2	$2p_{3/2}$	1.23104	1.21723	0.76922
	-3	$2d_{5/2}$	1.29851	1.28329	0.69221
	-4	$2f_{7/2}$	1.34844	1.33250	0.59863
			spin symmetry		
			$\nu = 0$	$\nu = 0.1$	$\nu = 1.0$
n	κ	state	$E > 0$	$E > 0$	$E > 0$
0	-2	$0p_{3/2}$	0.49589	0.57269	0.97850
	-3	$0d_{5/2}$	1.54413	1.55360	1.58112
	-4	$0f_{7/2}$	1.92856	1.92117	1.86174
	-5	$0g_{9/2}$	2.08474	2.07607	2.01207
1	-2	$1p_{3/2}$	—	—	0.17028
	-3	$1d_{5/2}$	1.21217	1.26027	1.48195
	-4	$1f_{7/2}$	1.99593	1.99371	1.96986
	-5	$1f_{9/2}$	2.19280	2.19280	2.17118

TABLE II: Energy eigenvalues in units of a.u. for the case where tensor interaction chosen as a Coulomb-like potential ($\eta_1 = \eta_2 = 2.5, M = 5, \nu = 0.5, \beta = 0.5$).

pseudospin symmetry				spin symmetry			
n	κ	state	$E < 0$	n	κ	state	$E > 0$
1	-1	$1s_{1/2}$	2.86397	0	-2	$0p_{3/2}$	—
	-2	$1p_{3/2}$	3.36173		-3	$0d_{5/2}$	0.86904
	-3	$1d_{5/2}$	3.57271		-4	$0f_{7/2}$	2.18996
	-4	$1f_{7/2}$	3.67004		-5	$0g_{9/2}$	2.78225
2	-1	$2s_{1/2}$	3.40594	1	-2	$1p_{3/2}$	—
	-2	$2p_{3/2}$	3.62177		-3	$1d_{5/2}$	—
	-3	$2d_{5/2}$	3.70932		-4	$1f_{7/2}$	0.03410
	-4	$2f_{7/2}$	3.74219		-5	$1f_{9/2}$	1.59973